

Distant total irregularity strength of graphs via random vertex ordering

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Abstract

Let $c : V \cup E \rightarrow \{1, 2, \dots, k\}$ be a (not necessarily proper) total colouring of a graph $G = (V, E)$ with maximum degree Δ . Two vertices $u, v \in V$ are *sum distinguished* if they differ with respect to sums of their incident colours, i.e. $c(u) + \sum_{e \ni u} c(e) \neq c(v) + \sum_{e \ni v} c(e)$. The least integer k admitting such colouring c under which every $u, v \in V$ at distance $1 \leq d(u, v) \leq r$ in G are sum distinguished is denoted by $\text{ts}_r(G)$. Such graph invariants link the concept of the total vertex irregularity strength of graphs with so called 1-2-Conjecture, whose concern is the case of $r = 1$. Within this paper we combine probabilistic approach with purely combinatorial one in order to prove that $\text{ts}_r(G) \leq (2 + o(1))\Delta^{r-1}$ for every integer $r \geq 2$ and each graph G , thus improving the previously best result: $\text{ts}_r(G) \leq 3\Delta^{r-1}$.

Keywords: total vertex irregularity strength of a graph, 1-2 Conjecture, r -distant total irregularity strength of a graph

1. Introduction

The cornerstone of the field of vertex distinguishing graph colourings is the graph invariant called *irregularity strength*. For a graph $G = (V, E)$ it is usually denoted by $s(G)$ and can be defined as the least integer k so that we may construct an irregular multigraph, i.e. a multigraph with pairwise distinct degrees of all vertices, of G by multiplying its edges, each at most k times (including the original one), see [8]. This study thus originated from the basic fact that no graph G with more than one vertex is irregular itself and related research on possible alternative definitions of an irregular graph, see e.g. [7]. Equivalently, $s(G)$ is also defined as the least k so that there exists an edge colouring $c : E \rightarrow \{1, 2, \dots, k\}$ such that for every pair $u, v \in V$, $u \neq v$, the sum of colours incident with u is distinct from the sum of colours incident with v . Note that $s(G)$ exists only for graphs without isolated edges and with at most one isolated

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vertex. It is known that $s(G) \leq n - 1$, where $n = |V|$, for all such graphs, except for K_3 , see [1, 21]. This tight upper bound can however be improved in the case of graphs with minimum degree $\delta \geq 1$ to $s(G) \leq 6\lceil \frac{n}{\delta} \rceil$ (what yields a better result whenever $\delta > 12$ and for $\delta \in [7, 12]$ if n is larger than a small constant dependent on δ), see [16], and to $s(G) \leq (4 + o(1))\frac{n}{\delta} + 4$ for graphs with $\delta \geq n^{0.5} \ln n$, see [19]. Many interesting results, concepts and open problems concerning this graph invariant can also be found e.g. in [6, 9, 10, 11, 12, 13, 16, 18, 25], and many others.

In [5], Bača et al. introduced a total version of the concept above. Given any graph $G = (V, E)$ and a (not necessarily proper) total colouring $c : V \cup E \rightarrow \{1, 2, \dots, k\}$, let

$$w_c(v) := c(v) + \sum_{u \in N(v)} c(uv) \quad (1)$$

denote the *weight* of any vertex $v \in V$, which shall also be called the *sum at v* and denoted simply by $w(v)$ in cases when c is unambiguous from context. The least k for which there exists such colouring with $w(u) \neq w(v)$ for every $u, v \in V$, $u \neq v$, is called the *total vertex irregularity strength* of G and denoted by $\text{tvs}(G)$. In [5], among others, it was proved that for every graph G with n vertices, $\lceil \frac{n+\delta}{\Delta+1} \rceil \leq \text{tvs}(G) \leq n + \Delta - 2\delta + 1$. Up to know the best upper bounds (for graphs with $\delta > 3$) assert that $\text{tvs}(G) \leq 3\lceil \frac{n}{\delta} \rceil + 1$, see [3], and $\text{tvs}(G) \leq (2 + o(1))\frac{n}{\delta} + 4$ for $\delta \geq n^{0.5} \ln n$, see [20]. Many other results e.g. for particular graph families can also be found in [4, 22, 26, 29] and other papers.

In this article we consider a distant generalization of $\text{tvs}(G)$ from [23], motivated among others by the study on distant chromatic numbers, see e.g. [17] for a survey concerning these. For any positive integer r , two distinct vertices at distance at most r in G shall be called *r -neighbours*. We denote by $N^r(v)$ the set of all r -neighbours of any $v \in V$ in G , and set $d^r(v) = |N^r(v)|$. The least integer k for which there exists a total colouring $c : V \cup E \rightarrow \{1, 2, \dots, k\}$ such that there are no r -neighbours u, v in G which are *in conflict*, i.e. with $w(u) = w(v)$ (cf. (1)), we call the *r -distant total irregularity strength* of G , and denote by $\text{ts}_r(G)$. It is known that $\text{ts}_r(G) \leq 3\Delta^{r-1}$ for every graph G , see [23], also for a comment implying that a general upper bound for $\text{ts}_r(G)$ cannot be (much) smaller than Δ^{r-1} . In this paper we combine the probabilistic method with algorithmic approach similar to those in e.g. [3, 15, 20, 23] to prove that in fact $\text{ts}_r(G) \leq (2 + o(1))\Delta^{r-1}$ (for $r \geq 2$).

Theorem 1. *For every integer $r \geq 2$ there exists a constant Δ_0 such that for each graph G with maximum degree $\Delta \geq \Delta_0$,*

$$\text{ts}_r(G) \leq 2\Delta^{r-1} + 3\Delta^{r-\frac{4}{3}} \ln^2 \Delta + 4, \quad (2)$$

hence

$$\text{ts}_r(G) \leq (2 + o(1))\Delta^{r-1}$$

for all graphs.

It is also worth mentioning that the case of $r = 1$ was introduced and considered separately in [27], where the well known 1-2-Conjecture concerning this invariant was introduced. It is known that $\text{ts}_1(G) \leq 3$ for all graphs, see Theorem 2.8 in [15], even in case of a natural list generalization of the problem, see [30], though it is believed that the upper bound of 2 should make the optimal general upper bound in both cases, see [27, 28, 31].

We also refer a reader to [24] to see an improvement of a similar probabilistic flavor for the upper bound from [23] on the correspondent of $\text{ts}_r(G)$ concerning the case of *edge* colourings exclusively.

2. Probabilistic Tools

We shall use probabilistic approach in the first part of the proof of Theorem 1, basing on the Lovász Local Lemma, see e.g. [2], combined with the Chernoff Bound, see e.g. [14] (Th. 2.1, page 26). We recall these below.

Theorem 2 (The Local Lemma). *Let A_1, A_2, \dots, A_n be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of all the other events A_j but at most D , and that $\Pr(A_i) \leq p$ for all $1 \leq i \leq n$. If*

$$ep(D+1) \leq 1,$$

then $\Pr(\bigcap_{i=1}^n \overline{A_i}) > 0$.

Theorem 3 (Chernoff Bound). *For any $0 \leq t \leq np$,*

$$\Pr(\text{BIN}(n, p) > np + t) < e^{-\frac{t^2}{3np}} \quad \text{and} \quad \Pr(\text{BIN}(n, p) < np - t) < e^{-\frac{t^2}{2np}} \leq e^{-\frac{t^2}{3np}}$$

where $\text{BIN}(n, p)$ is the sum of n independent Bernoulli variables, each equal to 1 with probability p and 0 otherwise.

Note that if X is a random variable with binomial distribution $\text{BIN}(n, p)$ where $n \leq k$, then we may still apply the Chernoff Bound above, even if we do not know the exact value of n , to prove that $\Pr(X > kp + t) < e^{-\frac{t^2}{3kp}}$ (for $t \leq \lfloor k \rfloor p$).

3. Proof of Theorem 1

Fix an integer $r \geq 2$. Within our proof we shall not specify Δ_0 . Instead, we shall assume that $G = (V, E)$ is a graph with sufficiently large maximum degree Δ , i.e. large enough so that all inequalities below are fulfilled.

We first partition V into a subset of vertices with relatively small degrees and a subset of those with big degrees:

$$\begin{aligned} S &= \left\{ u \in V : d(u) \leq \Delta^{\frac{2}{3}} \right\}; \\ B &= \left\{ u \in V : d(u) > \Delta^{\frac{2}{3}} \right\}; \end{aligned}$$

Moreover, for every $v \in V$, we denote: $S(v) = N(v) \cap S$, $s(v) = |S(v)|$, $B(v) = N(v) \cap B$, $b(v) = |B(v)|$.

Now we randomly order the vertices of V into a sequence. For this goal, associate with every vertex $v \in V$ a random variable $X_v \sim U[0, 1]$ having the uniform distribution on $[0, 1]$ where all these random variables X_v , $v \in V$ are independent, or in other words pick a (real) number uniformly at random from the interval $[0, 1]$ and associate it with v for every $v \in V$. Note that with probability one all these numbers are pairwise distinct.

In such a case, these independent random variables uniquely define a natural ordering v_1, v_2, \dots, v_n of the vertices in V where $X_{v_i} < X_{v_j}$ if and only if $1 \leq i < j \leq n$.

For every vertex $v \in V$, any its neighbour or r -neighbour u which precedes v in the obtained ordering of the elements of V shall be called a *backward neighbour* or r -*neighbour*, resp., of v . Analogously, the remaining ones shall be called *forward neighbours* or r -*neighbours*, resp., of v , while the edges joining v with its forward or backward neighbours shall be referred to as *forward* or *backward edges*, resp., as well. Also, for any subset $W \subset V$, let $N_-(v)$, $N_-^r(v)$, $N_W^r(v)$ denote the sets of all backward neighbours, backward r -neighbours and r -neighbours in W of v , respectively. Set $d_-^r(v) = |N_-^r(v)|$, $d_W^r(v) = |N_W^r(v)|$, and let $b_-(v)$ denote the number of backward neighbours of v which belong to $B(v)$.

Denote $D(v) = \sum_{u \in N(v)} d(u)$ and note that $d^r(v) \leq D(v)\Delta^{r-2} \leq d(v)\Delta^{r-1}$.

Let us also partition V into a subset I consisting of initial vertices of the obtained sequence and the remaining part R :

$$\begin{aligned} I &= \left\{ v : X_v < \frac{\ln \Delta}{\Delta^{\frac{1}{3}}} \right\}; \\ R &= \left\{ v : X_v \geq \frac{\ln \Delta}{\Delta^{\frac{1}{3}}} \right\}. \end{aligned}$$

Lemma 4. *With positive probability, the obtained ordering has the following features for every vertex v in G with $b(v) \geq \Delta^{\frac{1}{3}} \ln \Delta$:*

$$F_1: d_I^r(v) \leq 2d(v)\Delta^{r-\frac{4}{3}} \ln \Delta;$$

$$F_2: \text{if } v \in R, \text{ then: } b_-(v) \geq X_v b(v) - \sqrt{X_v b(v)} \ln \Delta;$$

$$F_3: \text{if } v \in R, \text{ then: } d_-^r(v) \leq X_v D(v)\Delta^{r-2} + \sqrt{X_v D(v)\Delta^{r-2}} \ln \Delta.$$

PROOF. For every vertex $v \in V$ of degree d in G and with $b(v) \geq \Delta^{\frac{1}{3}} \ln \Delta$ (hence also $d \geq \Delta^{\frac{1}{3}} \ln \Delta$), let $A_{v,1}$ denote the event that $d_I^r(v) > 2d\Delta^{r-\frac{4}{3}} \ln \Delta$, let $A_{v,2}$ be the event that v belongs to R and $b_-(v) < X_v b(v) - \sqrt{X_v b(v)} \ln \Delta$, and let $A_{v,3}$ denote the event that v belongs to R and $d_-^r(v) > X_v D(v)\Delta^{r-2} + \sqrt{X_v D(v)\Delta^{r-2}} \ln \Delta$.

As $|N^r(v)| \leq d\Delta^{r-1}$ and for each $u \in N^r(v)$, the probability that u belongs to I equals $\frac{\ln \Delta}{\Delta^{\frac{1}{3}}}$, then by the Chernoff Bound (and the comment below it),

$$\begin{aligned} \Pr(A_{v,1}) &\leq \Pr\left(d_I^r(v) > d\Delta^{r-\frac{4}{3}} \ln \Delta + \sqrt{d\Delta^{r-\frac{4}{3}} \ln \Delta \ln \Delta}\right) \\ &< e^{-\frac{d\Delta^{r-\frac{4}{3}} \ln^2 \Delta}{3d\Delta^{r-\frac{4}{3}} \ln \Delta}} = \Delta^{-\frac{\ln \Delta}{3}} < \frac{1}{\Delta^{3r}}. \end{aligned} \tag{3}$$

Subsequently note that for any $x \in [0, 1]$:

$$\begin{aligned} &\Pr(b_-(v) < X_v b(v) - \sqrt{X_v b(v)} \ln \Delta | X_v = x) \\ &= \Pr(\text{BIN}(b(v), x) < xb(v) - \sqrt{xb(v)} \ln \Delta) \\ &< \frac{1}{\Delta^{3r}}, \end{aligned}$$

where the last inequality follows by the Chernoff Bound if $\sqrt{xb(v)} \ln \Delta \leq xb(v)$, while it is trivial otherwise. Hence,

$$\Pr(A_{v,2}) \leq \Pr(b_-(v) < X_v b(v) - \sqrt{X_v b(v)} \ln \Delta) \leq \int_0^1 \frac{1}{\Delta^{3r}} dx = \frac{1}{\Delta^{3r}}. \quad (4)$$

For the sake of analyzing $A_{v,3}$, note now first that for $x \in [0, \frac{\ln \Delta}{\Delta^{\frac{1}{3}}})$,

$$\Pr(d_-(v) > X_v D(v) \Delta^{r-2} + \sqrt{X_v D(v) \Delta^{r-2}} \ln \Delta \wedge v \in R | X_v = x) = 0. \quad (5)$$

On the other hand, analogously as above, for $x \in [\frac{\ln \Delta}{\Delta^{\frac{1}{3}}}, 1]$:

$$\begin{aligned} & \Pr(d_-(v) > X_v D(v) \Delta^{r-2} + \sqrt{X_v D(v) \Delta^{r-2}} \ln \Delta \wedge v \in R | X_v = x) \\ & \leq \Pr(\text{BIN}(D(v) \Delta^{r-2}, x) > x D(v) \Delta^{r-2} + \sqrt{x D(v) \Delta^{r-2}} \ln \Delta) \\ & < \frac{1}{\Delta^{3r}}, \end{aligned} \quad (6)$$

where the last inequality follows by the Chernoff Bound, as $x \geq \frac{\ln \Delta}{\Delta^{\frac{1}{3}}}$ and $b(v) \geq \Delta^{\frac{1}{3}} \ln \Delta$ (where $D(v) \geq b(v) \Delta^{\frac{2}{3}}$) imply that $\sqrt{x D(v) \Delta^{r-2}} \ln \Delta \leq x D(v) \Delta^{r-2}$. Hence, by (5) and (6),

$$\Pr(A_{v,3}) \leq \int_0^1 \frac{1}{\Delta^{3r}} dx = \frac{1}{\Delta^{3r}}. \quad (7)$$

Note that each event $A_{v,i}$ is mutually independent of all other events except those $A_{u,j}$ with u at distance at most $2r$ from v , $i, j \in \{1, 2, 3\}$, i.e., at most $3\Delta^{2r} + 2$ events. Thus, as by (3), (4) and (7), the probability of each such event is bounded from above by Δ^{-3r} , by the Lovász Local Lemma, with positive probability none of the events $A_{v,i}$ with $v \in V$ (and $b(v) \geq \Delta^{\frac{1}{3}} \ln \Delta$) and $i \in \{1, 2, 3\}$ appears. \square

Let v_1, v_2, \dots, v_n be the ordering of the vertices of V guaranteed by Lemma 4. Set

$$K = \Delta^{r-1} + \lceil \Delta^{r-\frac{4}{3}} \ln^2 \Delta \rceil \quad \text{and} \quad k = \lceil \Delta^{r-\frac{4}{3}} \ln^2 \Delta \rceil,$$

and assign initial colour 1 to all the vertices and initial colour $K + 1$ to all the edges of G . We shall construct our final colouring $f : V \cup E \rightarrow \{1, 2, \dots, 2K + k + 1\}$ using an algorithm within which we shall be analyzing the consecutive vertices in the ordering (starting from v_1). Denote by $c_t(a)$ the contemporary colour of every $a \in V \cup E$ at every stage of the ongoing algorithm (hence initially $c_t(v) = 1$ and $c_t(e) = K + 1$ for every $v \in V$ and $e \in E$). The final target sum of every vertex $v \in V$, $w_f(v)$, shall be chosen the moment v is analyzed. For every $v \in V$, ever since $w_f(v)$ is chosen, we shall require so that

$$0 \leq w_f(v) - w_{c_t}(v) \leq K. \quad (8)$$

We shall admit at most two alterations of the colour for every edge in E - only when any of its ends is being analyzed (vertex colours shall be adjusted at the end of the algorithm). For every currently analyzed vertex v and its neighbour $u \in N(v)$, we admit the following alterations of the colour of $e = uv$ (the moment v is analyzed):

- adding $0, 1, \dots, K-1$ or K if e is a forward edge of v , $v \in S$ and $u \in B$,
- adding $0, 1, \dots, k-1$ or k if e is a forward edge of v (and $v \in B$ or $u \in S$),
- adding $-K, -K+1, \dots, K-1$ or K if e is a backward edge of v and $u \in B$,
- adding $-k, -k+1, \dots, k-1$ or k if e is a backward edge of v and $u \in S$,

so that afterwards (8) is fulfilled for every vertex $u \in N_-(v)$ and for v (after processing all edges incident with v). Note that the admitted alterations guarantee that $c_t(e) \in \{1, 2, \dots, 2K+k+1\}$ for every $e \in E$ at every stage of the construction.

Suppose we are about to analyze a vertex $v = v_i$, $i \in \{1, 2, \dots, n\}$, and thus far all our requirements have been fulfilled. We shall show that in every case the admitted alterations on the edges incident with v provide us more options for $w_{c_t}(v)$ than there are backward r -neighbours of v , and hence one of this options can be fixed as $w_f(v)$ so that this value is distinct from every $w_f(u)$ already fixed for any $u \in N_-^r(v)$. Denote the degree of v by d , and assume that $d > 0$ (otherwise, we set $w_f(v) = 1$):

- If $v \in I$, $v \in B$ and $b(v) \geq \Delta^{\frac{1}{3}} \ln \Delta$, then the admitted alterations provide at least $dk \geq d\Delta^{r-\frac{4}{3}} \ln^2 \Delta$ available options for $w_{c_t}(v)$. As by F_1 (from Lemma 4), $|N_-^r(v)| \leq d_I^r(v) \leq 2d\Delta^{r-\frac{4}{3}} \ln \Delta < d\Delta^{r-\frac{4}{3}} \ln^2 \Delta$, at least one of these available options is distinct from all $w_f(u)$ with $u \in N_-^r(v)$.
- If $v \in S$, then the admitted alterations provide at least $s(v)k + b(v)K \geq s(v)\Delta^{r-\frac{4}{3}} \ln^2 \Delta + b(v)(\Delta^{r-1} + \Delta^{r-\frac{4}{3}} \ln^2 \Delta)$ available options for $w_{c_t}(v)$. On the other hand, $|N_-^r(v)| \leq d^r(v) \leq D(v)\Delta^{r-2} \leq (s(v)\Delta^{\frac{2}{3}} + b(v)\Delta)\Delta^{r-2}$, hence at least one of these available options is distinct from all $w_f(u)$ with $u \in N_-^r(v)$.
- If $v \in B$ and $b(v) < \Delta^{\frac{1}{3}} \ln \Delta$, then the admitted alterations provide at least $dk \geq d\Delta^{r-\frac{4}{3}} \ln^2 \Delta$ available options for $w_{c_t}(v)$. On the other hand, analogously as in the case above, $|N_-^r(v)| \leq d^r(v) \leq s(v)\Delta^{\frac{2}{3}}\Delta^{r-2} + b(v)\Delta^{r-1} < d\Delta^{r-\frac{4}{3}} + \Delta^{r-\frac{2}{3}} \ln \Delta < d\Delta^{r-\frac{4}{3}} \ln^2 \Delta$, as $v \in B$ implies that $d \geq \Delta^{\frac{2}{3}}$. We thus have at least one option available for v distinct from all $w_f(u)$ with $u \in N_-^r(v)$.
- If $v \in R$, $v \in B$ and $b(v) \geq \Delta^{\frac{1}{3}} \ln \Delta$, then by F_2 the number of available options for $w_{c_t}(v)$ via admitted alterations of colours of the edges incident with v is not smaller than:

$$\begin{aligned}
b_-(v)K + (d - b_-(v))k &\geq b_-(v)\Delta^{r-1} + d\Delta^{r-\frac{4}{3}} \ln^2 \Delta \\
&\geq (X_v b(v) - \sqrt{X_v b(v)} \ln \Delta) \Delta^{r-1} + d\Delta^{r-\frac{4}{3}} \ln^2 \Delta \\
&\geq X_v b(v) \Delta^{r-1} - \sqrt{d} \Delta^{r-1} \ln \Delta + d\Delta^{r-\frac{4}{3}} \ln^2 \Delta \\
&\geq X_v b(v) \Delta^{r-1} + d\Delta^{r-\frac{4}{3}} \ln^2 \Delta - d\Delta^{r-\frac{4}{3}} \ln \Delta
\end{aligned}$$

(where the last inequality follows by the fact that $d \geq \Delta^{\frac{2}{3}}$). This number is however greater than the number of backward r -neighbours of v , as by F_3 ,

$$\begin{aligned}
|N_-^r(v)| &\leq X_v D(v) \Delta^{r-2} + \sqrt{X_v D(v) \Delta^{r-2}} \ln \Delta \\
&\leq X_v (b(v) \Delta + s(v) \Delta^{\frac{2}{3}}) \Delta^{r-2} + \sqrt{d \Delta^{r-1}} \ln \Delta \\
&\leq X_v b(v) \Delta^{r-1} + d\Delta^{r-\frac{4}{3}} + d\Delta^{r-\frac{4}{3}} \ln \Delta.
\end{aligned}$$

Thus in all cases there is at least one available sum, say w^* , for v which is distinct from all $w_f(u)$ with $u \in N_-^r(v)$. We then set $w_f(v) = w^*$ and perform the admitted alterations on the edges incident with v so that $w_{c_t}(v) = w^*$ afterwards.

By our construction, after analyzing v_n , all $w_f(v_i)$ are fixed for $i = 1, \dots, n$ so that $w_f(u) \neq w_f(v)$ whenever u and v are r -neighbours in G and (8) holds for every $v \in V$. We then modify (if necessary) the colour of every vertex v by adding to it the integer $w_f(v) - w_{c_t}(v)$, completing the construction of the desired total colouring f of G (by setting $f(a) = c_t(a)$ for every $a \in V \cup E$ afterwards). Note that $1 \leq f(e) \leq 2K + k + 1$ for every $e \in E$ and, by (8), $1 \leq f(v) \leq K + 1$ for every $v \in V$, hence the thesis follows. \square

4. Remarks

We have put an effort to optimize the second order term from the upper bound in (2), up to a constant and a power in the logarithmic factor, which could still be slightly improved (at the cost of the clarity of presentation). Nevertheless, some multiplicative poly-logarithmic (in Δ) factor seems unavoidable in this term within our approach.

We conclude by posing a conjecture, which to our believes expresses a true asymptotically optimal upper bound for the investigated parameters.

Conjecture 5. *For every integer $r \geq 2$ and each graph G with maximum degree Δ ,*

$$\text{ts}_r(G) \leq (1 + o(1))\Delta^{r-1}.$$

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